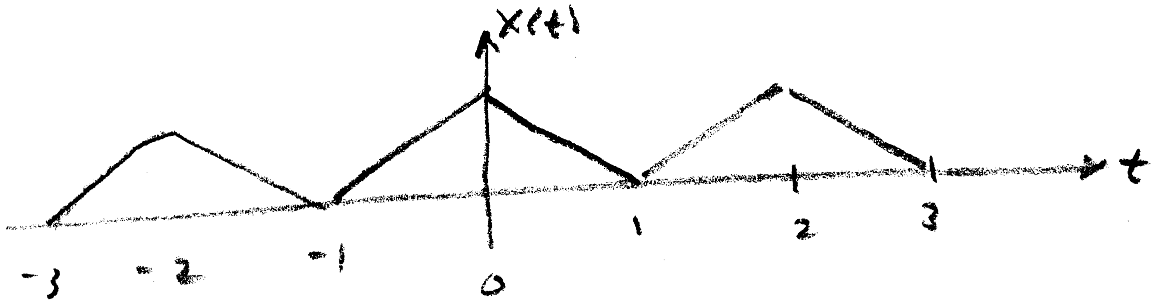
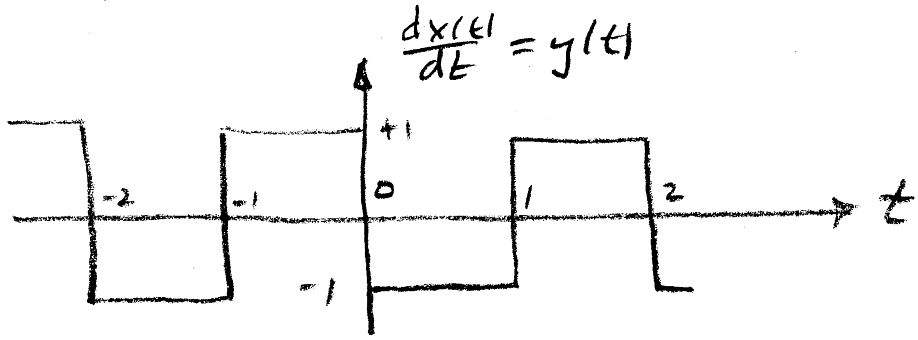
S2) The signal $x(t)$ is periodic and

$$x(t) = \begin{cases} t+1 & -1 < t < 0 \\ 1-t & 0 < t < +1 \end{cases}$$

with period $T = 2$ $\qquad \omega_0 = \frac{2\pi}{2} = \pi$



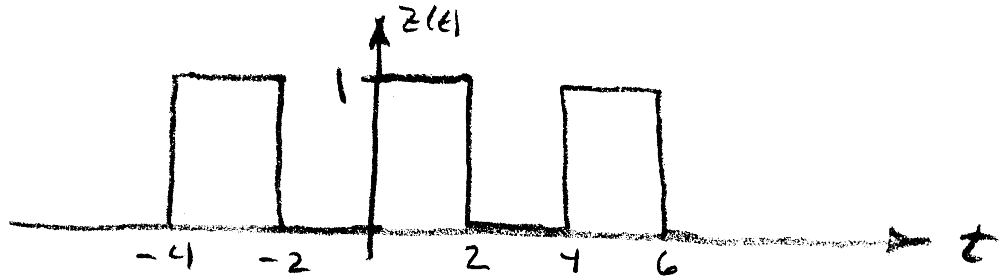
This signal has the following derivative



We can readily obtain the Fourier coefficients for $\frac{dx}{dt} = y(t)$. It has zero average value so

$$a_0 = 0$$

In class we obtained Fourier coefficients for the following square wave with period $T=4$



Its Fourier coefficients are

$$a_k = \begin{cases} \frac{1}{\pi j k} & k \text{ odd} \\ 0 & k \text{ even} \quad k \neq 0 \end{cases}$$

$$a_0 = \frac{1}{2}$$

The square wave $y(t)$ has zero average value, it is twice the magnitude of $z(t)$ and it is shifted forward in time by $\frac{1}{2}$ period. Hence, if b_k are the Fourier coefficients for $z(t)$ and since $T=2$ for $y(t)$

$$b_0 = 0$$

$$b_k = 2e^{j\omega_0 T} a_k = 2e^{j\frac{2\pi}{\cancel{T}}\cancel{T}} a_k = -2a_k$$

$$\therefore \quad b_k = \begin{cases} -\frac{2}{\pi j k} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

Now the signal we want is the integral of $y(t)$. Thus if c_k are the Fourier coefficients of $x(t)$

$$c_k = \frac{1}{jk\omega_0} b_k = \frac{1}{jk\pi} b_k = \begin{cases} -\frac{2}{(jk\pi)^2} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

$$= \begin{cases} \frac{2}{(k\pi)^2} & k \text{ odd} \\ 0 & k \text{ even} \quad k \neq 0 \end{cases}$$

For $k=0$ the coefficient is indeterminate with this equation. However we know that c_0 must be the average value of $x(t)$, which is $\frac{1}{2}$, so

$$c_0 = \frac{1}{2}$$

53)

(i)

$$a_k = \begin{cases} 1 & k=0 \\ j\left(\frac{1}{2}\right)^{|k|} & k \neq 0 \end{cases}$$

$$a_k^* = \begin{cases} 1 & k=0 \\ -j\left(\frac{1}{2}\right)^{|k|} & k \neq 0 \end{cases}$$

$$a_{-k} = \begin{cases} 1 & k=0 \\ j\left(\frac{1}{2}\right)^{|k|} & k \neq 0 \end{cases}$$

since $a_k^* \neq a_{-k}$ then $x(t)$ is not real

(ii) if

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

and

$$x(-t) \overset{FS}{\longleftrightarrow} b_k$$

Then

$$b_k = a_{-k} = \begin{cases} 1 & k=0 \\ j\left(\frac{1}{2}\right)^{k} & k \neq 0 \end{cases}$$

thus $b_k = a_k$

so $x(-t) = x(t)$ and $x(t)$ is even

5-3) (iii)

if

$$x(t) \overset{F.S.}{\longleftrightarrow} a_k$$

and

$$y(t) = \frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} b_k$$

then

$$b_k = jk\omega_0 a_k = \begin{cases} 0 & k=0 \\ -k\omega_0 \left(\frac{1}{2}\right)^{|k|} & k \neq 0 \end{cases}$$

and

$$y(-t) \overset{FS}{\longleftrightarrow} b_{-k} = \begin{cases} 0 & k=0 \\ k\omega_0 \left(\frac{1}{2}\right)^{|k|} & k \neq 0 \end{cases}$$

Thus, since $b_{-k} \neq b_k \Rightarrow y(-t) \neq y(t)$

and hence $y(t) = \frac{dx(t)}{dt}$ is not even.

S4)

i) $x(t) = \cos(\pi t) = \dfrac{e^{j\pi t} + e^{-j\pi t}}{2} = \frac{1}{2}e^{j\pi t} + \frac{1}{2}e^{-j\pi t}$

$\omega_0 = \pi$

Fourier coefficients are

$a_0 = 0 \quad a_1 = \frac{1}{2} \quad a_{-1} = \frac{1}{2} \quad a_k = 0$ otherwise

$x(t) = \sin(\pi t) = \dfrac{e^{j\pi t} - e^{-j\pi t}}{2j} = \frac{1}{2j}e^{j\pi t} - \frac{1}{2j}e^{-j\pi t}$

$\omega_0 = \pi$

Fourier coefficients are

$b_0 = 0 \quad b_1 = \frac{1}{2j} \quad b_{-1} = -\frac{1}{2j} \quad b_k = 0$ otherwise

ii) If

$$z(t) = x(t) \cdot y(t) \xleftrightarrow{F.S.} c_k$$

then

$$c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

Since only b_1 and b_{-1} are non zero, the only non-zero terms in the sum are for: $k-l=+1$ and $k-l=-1$

also, since only a_1 and a_{-1} are non-zero the only terms in the sum that will be non-zero are $l=+1$ and $l=-1$

Start at $k=0$

$$c_0 = a_1 b_{-1} + a_{-1} b_1 = \frac{1}{2}\left(-\frac{1}{2j}\right) + \frac{1}{2}\left(\frac{1}{2j}\right) = 0$$

$$c_1 = a_1 \cancel{b_0}^{\to 0} + a_{-1} \cancel{b_2}^{\to 0} = 0$$

$$c_2 = a_1 b_1 + a_{-1} \cancel{b_3}^{\to 0} = \frac{1}{2}\left(\frac{1}{2j}\right) = \frac{1}{4j}$$

$$c_{-1} = a_1 \cancel{b_{-2}}^{\to 0} + a_{-1} \cancel{b_0}^{\to 0} = 0$$

$$c_{-2} = a_1 b_{-3} + a_{-1} b_{-1} = \frac{1}{2}\left(-\frac{1}{2j}\right) = -\frac{1}{4j}$$

all other c_ks $= 0$

$$\therefore z(t) = \frac{1}{4j} e^{j2\pi t} - \frac{1}{4j} e^{-j2\pi t} = \frac{1}{2}\sin(2\pi t)$$